Enrollment No: ______________________ Exam Seat No:______________________

C.U.SHAH UNIVERSITY
Winter Examination -2021

Subject Name: Functional Analysis

Subject Code: 5SC03FUA1 **Branch: M.Sc. (Mathematics)**

Semester: 3 **Date: 15/12/2021** **Time: 02:30 To 05:30** **Marks: 70**

Instructions:
(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **[07]**

a. State Minkowski's inequality. (01)

b. For normed linear space X,show that $|\|x\| - \|y\|| \leq \|x - y\|$, $x, y \in X$. (02)

c. Show that $(R^2, \|\cdot\|_1)$ is not strictly convex. (02)

d. Define: Banach space. Give one example of normed linear space which is not a Banach space. (02)

Q-2 **Attempt all questions** **[14]**

a. Let $a_j, b_j \in K$ $(j = 1,2, \ldots n)$. Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then prove that $\sum_{j=1}^{n} |a_j b_j| \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}}$. (06)

b. Let X, Y be two norm linear space and $F: X \to Y$ be a linear map. If F is bounded on $\overline{U}(0, r)$ for some $r > 0$ then show that F is continuous on X. (05)

c. Let X, Y be normed linear space and $f: X \to Y$ be a linear map .If f is homeomorphism then show that $\exists\ \alpha, \beta > 0$ such that (03)

$$\beta\|x\| \leq \|f(x)\| \leq \alpha\|x\|, \forall x \in X.$$

OR

Q-2 **Attempt all questions** **[14]**

a. Let X, Y be normed linear space and $f: X \to Y$ be a linear map. Then show that f is continuous if and only if $Z(f)$ is closed in X and the linear map $\tilde{f}: X/Z(f) \to Y$ is continuous . (06)

b. State and prove Riesz Lemma. (05)

c. Let X, Y be Banach space and $F \in BL(X, Y)$.Prove that if F is bijective then $F^{-1} \in BL(X, Y)$. (03)

Q-3 **Attempt all questions** **[14]**

a. Let X, Y be normed linear space and $X \neq \{0\}$. Prove that Y be a Banach space if and only if $BL(X, Y)$ is a Banach space. (06)

	b.	Show that a Banach space can't have denumerable basis.	(05)
	c.	Let X be a normed linear space and $f \in X'$. Let $a \in X$ with $f(a) = 1$ then show that $U(a,r) \cap Z(f) = \emptyset$ if and only if $\|f\| \leq \frac{1}{r}$ for some $r > 0$.	(03)

OR

Q-3		**Attempt all questions**	**[14]**
	a.	Let X be a normed linear space over K and E_1, E_2 be nonempty, disjoint subsets of X where E_1 is open. Then prove that there exists a real hyper plane which separates E_1 and E_2 in the following sense. For some $f \in X'$ and $t \in R$, $Ref(x) < t \leq Re(y), \forall x \in E_1, y \in E_2$.	(07)
	b.	Let X be a normed linear space. Let $a \in X$ with $a \neq 0$ then prove that there exists $f \in X'$ such that $f(a) \neq 0$ and $\|f(a)\| = \|a\|$. Further, $\|a\| = \sup\{ \lvert f(a) \rvert : f \in X', \|f\| \leq 1\}$	(05)
	c.	Show that closure of a convex set in a normed linear space is convex set.	(02)

SECTION – II

Q-4		**Attempt the Following questions**	**[07]**
	a.	Determine whether the statement is True or False :Every Hilbert space has Schauder Basis.	(01)
	b.	Define: Transpose of bounded linear map.	(01)
	c.	State open mapping theorem.	(01)
	d.	Prove that weak limit of a weakly convergent sequence is unique.	(02)
	e.	Let P be a projection on a normed linear space X.If both $Z(P)$ and $R(P)$ are closed in X,then show that P is a closed map.	(02)
Q-5		**Attempt all questions**	**[14]**
	a.	Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For a fixed $y \in l^q$, define $f_y : l^p \rightarrow K$ by $f_y(x) = \sum_{j=1}^{\infty} x(j)y(j)$ then prove that $f_y \in (l^p)'$ and $\|f_y\| = \|y\|_q$. Also show that the mapping $F: l^q \rightarrow (l^p)'$ is a linear isometry.	(09)
	b.	Let X be a Banach space and $A \in BL(X)$ then show that $\sigma(A) = \sigma_a(A) \cup \sigma_e(A') = \sigma(A')$.	(05)

OR

Q-5		**Attempt all questions**	**[14]**
	a.	State and Prove Uniform Bounded Principle.	(06)
	b.	Show that completeness of normed linear spaces can't be dropped in open mapping theorem.	(05)
	c.	Let X, Y, Z be normed linear spaces. Then prove the following: i) $(F_1 + F_2)' = F_1' + F_2'$, where $F_1, F_2 \in BL(X,Y)$ ii) $(G \circ F)' = F' \circ G'$, where $F \in BL(X,Y)$ and $G \in BL(Y,Z)$	(03)
Q-6		**Attempt all questions**	**[14]**
	a.	Let X, Y be Banach space and $F: X \rightarrow Y$ be a linear map.Prove that if F is closed map then F is continuous.	(09)
	b.	Let Z be a closed subspace of a normed linear space X. Let $Q: X \rightarrow X/Z$ defined by $Q(x) = x + Z$. Show that Q is continuous and open. State the result you use.	(05)

OR

Q-6 **Attempt all Questions** **[14]**

a. Let X be a normed linear space and $A \in BL(X)$be a finite rank operator, then prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$. (07)

b. Define :Absolutely Summable Series. Let X be a normed linear space.Then show that X is complete if and only if every absolutely summable series in X is summable in X. (07)